Correlations in Complex Systems

Significant Memory Effects Typically Cause Long Time Correlations in Complex Systems

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Article Outline

Glossary

Correlation A correlation describes the degree of relationship between two or more variables. The correlations are viewed due to the impact of random factors and can be characterized by the methods of probability theory.

Correlation function The correlation function (abbreviated, as CF) represents the quantitative measure for the compact description of the wide classes of correlation in the complex systems (CS). The correlation function of two variables in statistical mechanics provides a measure of the mutual order existing between them. It quantifies the way random variables at different positions are correlated. For example in a spin system, it is the thermal average of the scalar product of the spins at two lattice points over all possible orderings.

Memory effects in stochastic processes through correlations Memory effects (abbreviated, as ME) appear at a more detailed level of statistical description of correlation in the hierarchical manner. ME reflect the complicated or hidden character of creation, the propagation and the decay of correlation. ME are produced by inherent interactions and statistical after-effects in CS. For the statistical systems ME are induced by contracted description of the evolution of the dynamic variables of a CS.

Memory functions Memory functions describe mutual interrelations between the rates of change of random variables on different levels of the statistical description. The role of memory has its roots in the natural sciences since 1906 when the famous Russian mathematician Markov wrote his first paper in the theory of Markov Random Processes. The theory is based on the notion of the instant loss of memory from the prehistory (memoryless property) of random processes.

Information measures of statistical memory in complex systems From the physical point of view time scales of correlation and memory cannot be treated as arbitrary. Therefore, one can introduce some statistical quantifiers for the quantitative comparison of these time scales. They are dimensionless and possess the statistical spectra on the different levels of the statistical description.

Definition of the Subject

As commonly used in probability theory and statistics, a correlation (also so called correlation coefficient), measures the strength and direction of a linear relationship between two random variables. In a more general sense, a correlation or co-relation reflects the deviation of two (or more) variables from mutual independence, although correlation does not imply causation. In this broad sense there are some quantifiers which measures the degree of correlation, suited to the nature of data. Increasing attention has been paid recently to the study of statistical memory effects in random processes that originate from nature by means of non-equilibrium statistical physics. The role of memory has its roots in natural sciences since 1906 when the famous Russian mathematician Markov wrote his first paper on the theory of Markov Random Processes (MRP) [1]. His theory is based on the notion of an instant loss of memory from the prehistory (memoryless property) of random processes. In contrast, there are an abundance of physical phenomena and processes which can be characterized by statistical memory effects: kinetic and relaxation processes in gases [2] and plasma [3], condensed matter physics (liquids [4], solids [5], and superconductivity [6]) astrophysics [7], nuclear physics [8], quantum [9] and classical [9] physics, to name only a few. At present, we have a whole toolbox available of statistical methods which can be efficiently used for the analysis of the memory effects occurring in diverse physical systems. Typical such schemes are Zwanzig–Mori's kinetic equations [10,11], generalized master equations and corresponding statistical quantifiers [12,13,14,15,16,17,18], Lee's recurrence relation method [19,20,21,22,23], the generalized Langevin equation (GLE) [24,25,26,27,28,29], etc.

Here we shall demonstrate that the presence of statistical memory effects is of salient importance for the functioning of the diverse natural complex systems. Particularly, it can imply that the presence of large memory times scales in the stochastic dynamics of discrete time series can characterize catastrophical (or pathological for live systems) violation of salutary dynamic states of CS. As an example, we will demonstrate here that the emergence of strong memory time scales in the chaotic behavior of complex systems (CS) is accompanied by the likely initiation and the existence of catastrophes and crises (Earthquakes, financial crises, cardiac and brain attack, etc.) in many CS and especially by the existence of pathological states (diseases and illness) in living systems.

Introduction

A common definition [30] of a correlation measure $\rho(X, Y)$ between two random variables X and Y with the mean values $E(X)$ and $E(Y)$, and fluctuations $\delta X = X - E(X)$ and $\delta Y = Y - E(Y)$, dispersions $\sigma_X^2 = E(\delta X^2) = E(X^2) - E(X)^2$ and $\sigma_Y^2 = E(\delta Y^2) = E(Y^2) - E(Y)^2$ is defined by:

$$\rho(X, Y) = \frac{E(\delta X\, \delta Y)}{\sigma_X\, \sigma_Y} ,$$

where E is the expected value of the variable. Therefore we can write

$$\rho(X, Y) = \frac{[E(XY) - E(X)\, E(Y)]}{(E(X^2) - E(X)^2)^{1/2}\, (E(Y^2) - E(Y)^2)^{1/2}} .$$

Here, a correlation can be defined only if both of the dispersions are finite and both of them are nonzero. Due to the Cauchy–Schwarz inequality, a correlation cannot exceed 1 in absolute value. Consequently, a correlation assumes it maximum at 1 in the case of an increasing linear relationship, or -1 in the case of a decreasing linear relationship, and some value in between in all other cases, indicating the degree of linear dependence between the variables. The closer the coefficient is either to -1 or 1, the stronger is the correlation between the variables. If the variables are independent then the correlation equals 0, but the converse is not true because the correlation coefficient detects only linear dependencies between two variables.

Since the absolute value of the sample correlation must be less than or equal to 1 the simple formula conveniently suggests a single-pass algorithm for calculating sample correlations. The square of the sample correlation coefficient, which is also known as the coefficient of determination, is the fraction of the variance in σ_x that is accounted for by a linear fit of x_i to σ_y. This is written

$$R_{xy}^2 = 1 - \frac{\sigma_{y|x}^2}{\sigma_y^2} ,$$

where $\sigma_{y|x}^2$ denotes the square of the error of a linear regression of x_i on y_i in the equation $y = a + bx$,

$$\sigma_{y|x}^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

and σ_y^2 denotes just the dispersion of y.

Note that since the sample correlation coefficient is symmetric in x_i and y_i , we will obtain the same value for a fit to y_i:

$$R_{xy}^2 = 1 - \frac{\sigma_{x|y}^2}{\sigma_x^2} .$$

This equation also gives an intuitive idea of the correlation coefficient for random (vector) variables of higher dimension. Just as the above described sample correlation coefficient is the fraction of variance accounted for by the fit of a 1-dimensional linear submanifold to a set of 2-dimensional vectors (x_i, y_i), so we can define a correlation coefficient for a fit of an m-dimensional linear submanifold to a set of n-dimensional vectors. For example, if we fit a plane $z = a + bx + cy$ to a set of data (x_i, y_i, z_i) then the correlation coefficient of z to x and y is

$$R^2 = 1 - \frac{\sigma_{z|xy}^2}{\sigma_z^2} .$$

Correlation and Memory in Discrete Non-Markov Stochastic Processes

Here we present a non-Markov approach [31,32] for the study of long-time correlations in chaotic long-time dynamics of CS. For example, let the variable x_i be defined as the R-R interval or the time distance between nearest, so called R peaks occurring in a human electrocardiogram (ECG). The generalization will consist in taking into account non-stationarity of stochastic processes and its further applications to the analysis of the heart-rate-variability.

We should bear in mind, that one of the key moments of the spectral approach in the analysis of stochastic processes consists in the use of normalized time correlation function (TCF)

$$a_0(t) = \frac{\langle\langle \mathbf{A}(T)\,\mathbf{A}(T+t)\rangle\rangle}{\langle \mathbf{A}(T)^2\rangle} \,. \tag{1}$$

Here the time T indicates the beginning of a time serial, $\mathbf{A}(t)$ is a state vector of a complex system as defined below in Eq. (5) at t, $|\mathbf{A}(t)|$ is the length of vector $\mathbf{A}(t)$, the double angular brackets indicate a scalar product of vectors and an ensemble averaging. The ensemble averaging is, of course needed in Eq. (1) when correlation and other characteristic functions are constructed. The average and scalar product becomes equivalent when a vector is composed of elements from a discrete-time sampling, as done later. Here a continuous formalism is discussed for convenience. However further, since Sect. "Correlation and Memory in Discrete Non-Markov Stochastic Processes" we shall consider only a case of discrete processes.

The above-stated designation is true only for stationary systems. In a non-stationary case Eq. (1) is not true and should be changed. The concept of TCF can be generalized in case of discrete non-stationary sequence of signals. For this purpose the standard definition of the correlation coefficient in probability theory for the two random signals X and Y must be taken into account

$$\rho = \frac{\langle\langle \mathbf{XY}\rangle\rangle}{\sigma_X\,\sigma_Y}\,, \quad \sigma_X = \langle|\mathbf{X}|\rangle\,, \quad \sigma_Y = \langle|\mathbf{Y}|\rangle\,. \tag{2}$$

In Eq. (2) the multi-component vectors $\mathbf{X}$, $\mathbf{Y}$ are determined by fluctuations of signals x and y accordingly, σ_X^2, σ_Y^2 represent the dispersions of signals $\mathbf{x}$ and $\mathbf{y}$, and values $|\mathbf{X}|$, $|\mathbf{Y}|$ represent the lengths of vectors $\mathbf{X}$, $\mathbf{Y}$, correspondingly. Therefore, the function

$$a(T,t) = \frac{\langle\langle \mathbf{A}(T)\,\mathbf{A}(T+t)\rangle\rangle}{\langle|\mathbf{A}(T)|\rangle\,\langle|\mathbf{A}(T+t)|\rangle} \tag{3}$$

can serve as the generalization of the concept of TCF (1) for non-stationary processes $\mathbf{A}(T+t)$. The non-stationary TCF (3) obeys the conditions of the normalization and attenuation of correlation

$$a(T,0) = 1\,, \quad \lim_{t\to\infty} a(T,t) = 0\,.$$

Let us note, that in a real CS the second limit, typically, is not carried out due possible occurrence nonergodocity (meaning that a time average does not equal its ensemble average). According to the Eqs. (1) and (3) for the quantitative description of non-stationarity it is convenient to introduce a function of non-stationarity

$$\gamma(T,t) = \frac{\langle|\mathbf{A}(T+t)|\rangle}{\langle|\mathbf{A}(T)|\rangle} = \left\{\frac{\sigma^2(T+t)}{\sigma^2(T)}\right\}^{1/2}\,. \tag{4}$$

One can see that this function equals the ratio of the lengths of vectors of final and initial states. In case of stationary process the dispersion does not vary with the time (or its variation is very weak). Therefore the following relations

$$\sigma(T+t) = \sigma(T)\,, \quad \gamma(T,t) = 1 \tag{5}$$

hold true for the stationary process.

Due to the condition (5) the following function

$$\Gamma(T,t) = 1 - \gamma(T,t) \tag{6}$$

is suitable in providing a dynamic parameter of non-stationarity. This dynamic parameter can serve as a quantitative measure of non-stationarity of the process under investigation. According to Eqs. (4)–(6) it is reasonable to suggest the existence of three different elementary classes of non-stationarity

$$|\Gamma(T,t)| = |1-\gamma(T,t)| = \left\{\begin{array}{ll} \ll 1, & \text{weak non-stationarity} \\ \sim 1, & \text{intermediate non-stationarity} \\ \gg 1, & \text{strong non-stationarity} \end{array}\right\}\,. \tag{7}$$

The existence of dynamic parameter of non-stationarity makes it possible to determine , on-principle, the type of non-stationarity of the underlying process and to find its spectral characteristics from the experimental data base. We intend to use Eqs. (4), (6), (7) for the quantitative description of effects of non-stationarity in the investigated temporary series of R-R intervals of human ECG's for healthy people and patients after myocardial infarction (MI).

Statistical Theory of Non-Stationary Discrete Non-Markov Processes in Complex Systems TS2

Here we shall extend the original results of the statistical theory of discrete non-Markov processes in complex systems, developed recently in [31], for the case of non-stationary processes. The theory [31] is developed on the basis of first principles and represents a discrete finite-difference analogy for complex systems of well known Zwanzig–Mori's kinetic equations [10,11,12,13,14,15,16,17,18] used in the statistical physics of condensed matter.

We examine a discrete stochastic process $X(T+t)$, where $t = m\tau$

$$X = \{x(T), x(T+\tau), x(T+2\tau), \ldots, x(T+k\tau), \\ \ldots, x(T+(N-1)\tau)\}\,, \quad (8)$$

where T is the beginning of the time and τ is a discretization time. The normalized time correlation function (TCF)

$$a(t) = \frac{1}{(N-m)\,\sigma^2} \sum_{j=0}^{N-1-m} \delta x(T+j\tau)\,\delta x(T+(j+m)\,\tau) \quad (9)$$

yields a convenient measure to analyze the dynamic properties of complex systems. Herein, we used the variance σ^2, the fluctuation $\delta x(T+j\tau)$, which in terms of the the mean value $\langle x \rangle$ reads:

$$\delta x_j = \delta x(T+j\tau) = x(T+j\tau) - \langle x \rangle\,, \\ \sigma^2 = \frac{1}{(N-m)} \sum_{j=0}^{N-1-m} \{\delta x(T+j\tau)\}^2\,, \quad (10)$$

$$\langle x \rangle = \frac{1}{(N-m)} \sum_{j=0}^{N-1-m} x(T+j\tau)\,. \quad (11)$$

The discrete time t is given as $t = m\tau$.

In general, the mean value, the variance and TCF in (9), (10) and (11) is dependent on the numbers m and N. All indicated values cease to depend on numbers m and N for stationary processes when $m \ll N$. The definition of TCF in Eq. (9) is true only for stationary processes.

Next, we shall try to take into account this important dependence. With this purpose we shall form two k-dimensional vectors of state by the process (8):

$$\mathbf{A}_k^0 = (\delta x_0, \delta x_1, \delta x_2, \ldots, \delta x_{k-1})\,, \\ \mathbf{A}_{m+k}^m = (\delta x_m, \delta x_{m+1}, \delta x_{m+2}, \ldots, \delta x_{m+k-1})\,. \quad (12)$$

When a vector of a state is composed of elements from a discrete-time sampling, the average and scalar product in Eq. (1) become equivalent. In an Euclidean space of vectors of state (12) TCF $a(t)$

$$a(t) = \frac{\langle \mathbf{A}_{N-1-m}^0\, \mathbf{A}_{N-1}^m \rangle}{(N-m)\{\sigma(N-m)\}^2} = \frac{\langle \mathbf{A}_{N-1-m}^0\, \mathbf{A}_{N-1}^m \rangle}{|\mathbf{A}_{N-1-m}^0|^2} \quad (13)$$

describes the correlation of two different states of the system ($t = m\tau$). Here the brackets $\langle \ldots \rangle$ indicate the scalar product of the two vectors. The dimension dependence of the corresponding vectors is also taken into account in the variance $\sigma = \sigma(N-m)$. As a matter of fact TCF $a(t) = \cos\vartheta$, where ϑ is the angle between the two vectors from Eq. (12). Let's introduce a unit vector of dimension $(N-m)$ in the following way:

$$\mathbf{n} = \frac{\mathbf{A}_{N-1-m}^0}{\sqrt{(N-m)\sigma^2}}\,. \quad (14)$$

Then, the TCF $a(t)$ (9) is given by

$$a(t) = \langle \mathbf{n}(0)\, \mathbf{n}(t) \rangle\,. \quad (15)$$

From the above discussion it is evident that Eqs. (13)–(15) are true for the stationary processes only. In case of non-stationary processes it is necessary to redefine TCF, taking into account the non-stationarity in the variance σ^2 in a line with Eqs.(2)–(7). For this purpose we shall redefine a unit vector of the final state as following

$$\mathbf{n}(t) = \frac{\mathbf{A}_{N-1}^m(t)}{|\mathbf{A}_{N-1}^m(t)|}\,. \quad (16)$$

For non-stationary processes it is convenient to write the TCF as the scalar product of the two unit vectors of the initial and final states

$$a(t) = \langle \mathbf{n}(0)\, \mathbf{n}(t) \rangle = \frac{\langle \mathbf{A}_{N-1-m}^0(0)\, \mathbf{A}_{N-1}^m(t) \rangle}{|\mathbf{A}_{N-1-m}^0(0)|\,|\mathbf{A}_{N-1}^m(t)|}\,. \quad (17)$$

Now we shall turn to the the dynamics of a non-stationary stochastic process. The equation of motion of a the random process x_j can be w[illegible]n in a finite-difference form for $0 \le j \le N-1$ [15] TS3 in the following way

$$\frac{dx_j}{dt} \Rightarrow \frac{\Delta \delta x_j}{\Delta t} = \frac{\delta x_j(t+\tau) - \delta x_j(t)}{\tau}\,. \quad (18)$$

Then it is convenient to define the discrete evolution single step operator $\hat{U}$ as following:

$$x(T+(j+1)\,\tau) = \hat{U}(T+(j+1)\,\tau, T+j\tau)\, x(T+j\tau)\,. \quad (19)$$

In the case of stationary process we can rewrite the equation of motion (18) in a more simple form

$$\frac{\Delta \delta x_j}{\Delta t} = \tau^{-1}\big\{\hat{U}(\tau) - 1\big\}\, \delta x_j\,. \quad (20)$$

The invariance of the mean value $\langle x \rangle$ is taken into account in an Eq. (20)

$$\langle x \rangle = \hat{U}(\tau)\langle x \rangle\,, \quad \big\{\hat{U}(\tau) - 1\big\}\langle x \rangle = 0\,. \quad (21)$$

In case of a non-stationary process it is necessary to turn to the equation of motion for vector of the final state $\mathbf{A}_{m+k}^m(t)$ $(k = N-1-m)$

$$\frac{\Delta \mathbf{A}_{m+k}^m(t)}{\Delta t} = i\hat{L}(t,\tau)\, \mathbf{A}_{m+k}^m(t)\,, \quad (22)$$

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where Liouville's quasioperator is

$$\hat{L}(t,\tau) = (i\tau)^{-1}\{\hat{U}(t+\tau,t) - 1\}\,. \tag{23}$$

It is well known that, in general, a stochastic trajectory does not obey a linear equation, so the general evolution operator and Liouville's quasioperator should probably be non-linear. Furthermore, in statistical physics the Liouville's operator acts upon the probability densities of dynamical variables, as well upon the variables itself like in Mori's paper [12]. The evolution of the density would be indeed linear. But Mori used the Liouville operator in the quantum equation of motion in [12]. In line with Mori [12] Eqs. (20), (22) can be considered as formal and exact equations of the motion of a complex system.

Thus, due to the Eqs. (17), (22) and (23) we may take into account the non-stationarity of the stochastic process. Towards this goal let's introduce the linear projection operator in Euclidean space of the state vectors

$$\Pi\mathbf{A}(t) = \frac{\mathbf{A}(0)\rangle\langle\mathbf{A}(0)\,\mathbf{A}(t)\rangle}{|\mathbf{A}(0)|^2}\,, \quad \Pi = \frac{\mathbf{A}(0)\rangle\langle\mathbf{A}(0)}{\langle\mathbf{A}(0)\,\mathbf{A}(0)\rangle}\,, \tag{24}$$

where angular brackets in numerator present the boundaries of action for the scalar product.

For the analyzing the dynamics TS4 of the stochastic process $\mathbf{A}(t)$ the vector $\mathbf{A}_k^0(0)$ from (12) can be considered as a vector of the initial state $\mathbf{A}(0)$, and vector $\mathbf{A}_{m+k}^m(t)$ from (12) at value $m + k = N - 1$ can be considered as the vector of the final state $\mathbf{A}(t)$.

It is necessary to note that the projection operator (24) has the required property of idem-potency $\Pi^2 = \Pi$. The presence of operator Π allows one to introduce the mutually supplementary projection operator P:

$$P = 1 - \Pi\,, \quad P^2 = P\,, \quad \Pi P = P\Pi = 0\,. \tag{25}$$

It is necessary to remark, that both projectors Π and P are linear and can be recorded for the fulfillment of operations in the particular Euclidean space. Due to the property (17) and Eq. (4) it is easy to obtain the required TCF:

$$\begin{aligned}\Pi\mathbf{A}(t) &= \Pi\mathbf{A}_{m+k}^m(t)\\ &= \mathbf{A}_k^0(0)\langle\mathbf{n}_k^0(0)\,\mathbf{n}_{k+m}^m(t)\rangle\gamma_1(t)\\ &= \mathbf{A}_k^0(0)\,a(t)\,\gamma_1(t)\,,\\ \gamma_1(t) &= \frac{|\mathbf{A}_{m+k}^m(t)|}{|\mathbf{A}_m^0(0)|}\,.\end{aligned} \tag{26}$$

Therefore the projector Π generates a unit vector along the vector of the final state $\mathbf{A}(t)$ and makes its projection onto the initial state vector $\mathbf{A}(0)$.

The existence of a pair of two mutually supplementary projection operators Π and P allows one to carry out the splitting of Euclidean space of vectors $A(\mathbf{A}(0), \mathbf{A}(t) \in A)$ into a straight sum of two mutually supplementary subspaces in the following way

$$A = A' \dot{+} A''\,, \quad A' = \Pi A\,, \quad A'' = PA\,. \tag{27}$$

Substituting Eq. (27) in Eq. (23) we find Liouville's quasioperator $\hat{L}$ in a matrix form

$$\hat{L} = \hat{L}_{11} + \hat{L}_{12} + \hat{L}_{21} + \hat{L}_{22}\,, \tag{28}$$

where the matrix elements are introduced

$$\begin{aligned}&\hat{L}_{11} = \Pi\hat{L}\Pi\,, \quad \hat{L}_{12} = \Pi\hat{L}P\,,\\ &\hat{L}_{21} = P\hat{L}\Pi\,, \quad \hat{L}_{22} = P\hat{L}P\,.\end{aligned} \tag{29}$$

The Euclidean space of values of Liouville's quasioperator $W = \hat{L}A$ will be generated by the vectors $\mathbf{W}$ of dimension $k - 1$

$$\begin{aligned}&(\mathbf{W}(0) \in W\,, \ \mathbf{W}(t) \in W)\\ &W = W' \dot{+} W''\,, \quad W' = \Pi W\,, \quad W'' = PW\,.\end{aligned} \tag{30}$$

Matrix elements $\hat{L}_{ij}$ of the contracted description

$$\hat{L} = \begin{pmatrix}\hat{L}_{11} & \hat{L}_{12}\\ \hat{L}_{21} & \hat{L}_{22}\end{pmatrix} \tag{31}$$

are acting in the following way:

$\hat{L}_{11}$ – from a subspace A' to subspace W',
$\hat{L}_{12}$ – from A'' to W',
$\hat{L}_{21}$ – from W' to W'' and
$\hat{L}_{22}$ – from A'' to W''.

The projection operators Π and P provide the contracted description of the stochastic process. Splitting the dynamic Eq. (22) into two equations in the two mutually supplementary Euclidean subspaces (see, for example [11] TS5), we find

$$\frac{\Delta\mathbf{A}'(t)}{\Delta t} = i\hat{L}_{11}\,\mathbf{A}'(t) + i\hat{L}_{12}\,\mathbf{A}''(t)\,, \tag{32}$$

$$\frac{\Delta\mathbf{A}''(t)}{\Delta t} = i\hat{L}_{21}\,\mathbf{A}'(t) + i\hat{L}_{22}\,\mathbf{A}''(t)\,. \tag{33}$$

Following [31,32] it is necessary to eliminate first the irrelevant part $\mathbf{A}''(t)$ in order to simplify Liouville's Eq. (22) and then to write a closed equation for relevant part $\mathbf{A}'(t)$. According to [32] that can be achieved by a series of successive steps (for example, see Eqs. (32)–(36)

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in [32]). First a solution to Eq. (33) for the first step can be obtained in a form

$$\begin{aligned}\frac{\Delta \mathbf{A}''(t)}{\Delta t} &= \frac{\mathbf{A}''(t+\tau)-\mathbf{A}''(t)}{\tau}\\ &= i\hat{L}_{21}\,\mathbf{A}'(t) + i\hat{L}_{22}\,\mathbf{A}''(t)\,,\\ \mathbf{A}''(t+\tau) &= \mathbf{A}''(t) + i\tau\,\hat{L}_{21}\,\mathbf{A}'(t) + i\tau\,\hat{L}_{22}\,\mathbf{A}''(t)\\ &= \{1+i\tau\,\hat{L}_{22}\}\,\mathbf{A}''(t) + i\tau\,\hat{L}_{21}\,\mathbf{A}'(t)\\ &= U_{22}(t+\tau,t)\,\mathbf{A}''(t) + i\tau\,\hat{L}_{21}(t+\tau,t)\,\mathbf{A}'(t)\,.\end{aligned} \tag{34}$$

We next can derive a finite-difference kinetic equation of a non-Markov type for TCF $a(t=m\tau)$

$$\frac{\Delta a(t)}{\Delta t} = \lambda_1 a(t) - \tau\Lambda_1 \sum_{j=0}^{m-1} M_1(t-j\tau)\,a(j\tau)\,. \tag{35}$$

Here, λ_1 is a eigenvalue, Λ_1 is a relaxation parameter of Liouville's quasioperator $\hat{L}$

$$\begin{aligned}\lambda_1 &= i\,\frac{\langle \mathbf{A}_k^0(0)\,\hat{L}\,\mathbf{A}_k^0(0)\rangle}{|\mathbf{A}_k^0(0)|^2}\,,\\ \Lambda_1 &= \frac{\langle \mathbf{A}_k^0(0)\,\hat{L}_{12}\,\hat{L}_{21}\,\mathbf{A}_k^0(0)\rangle}{|\mathbf{A}_k^0(0)|^2} = \frac{\langle \mathbf{A}_k^0(0)\,\hat{L}^2\,\mathbf{A}_k^0(0)\rangle}{|\mathbf{A}_k^0(0)|^2}\,,\end{aligned} \tag{36}$$

The angular brackets indicate here a scalar product of new state vectors. Function $M_1(t-j\tau)$ on the right side of Eq. (35) represents a modified memory function (MF) of the first order

$$M_1(t-j\tau) = \frac{\gamma_1(t-j\tau)}{\gamma_1(t)}\,m_1(t-j\tau)\,. \tag{37}$$

For stationary processes the function $\gamma_1(t)$ approaches unity. Then the memory functions $M_1(t)$ and $m_1(t)$ coincide with each other. The latter equation is the first kinetic finite-difference equation for TCF. It is remarkable, that the non-Markovity, discretization and non-stationarity of stochastic process can be considered explicitly. Due to the presence of non-stationarity both in TCF and in the first memory function this equation generalizes our results recently obtained in [31].

Following the projection technique described above, we arrive at a chain of connected kinetic finite-difference equations of a non-Markov type for the normalized short memory functions $m_n(t)$ in Euclidean space of state vectors of dimension $(k-n)$ $(t=m\tau,\ n\geq 1)$

$$\begin{aligned}\frac{\Delta m_n(t)}{\Delta t} &= \lambda_{n+1}\,m_n(t) - \tau\Lambda_{n+1}\\ &\quad\times \sum_{j=0}^{m-1} m_{n+1}(j\tau)\,m_n(t-j\tau)\\ &\quad\times \left\{\frac{\gamma_{n+1}(j\tau)\gamma_{n+1}(t-j\tau)}{\gamma_n(t)}\right\}\,,\\ m_{n+1}(t) &= \frac{\langle \mathbf{W}_{n+1}(0)\,\mathbf{W}_{n+1}(t)\rangle}{|\mathbf{W}_{n+1}(0)||\mathbf{W}_{n+1}(t)|}\,,\end{aligned} \tag{38}$$

$$\gamma_n(j\tau) = \left\{\frac{|\mathbf{W}_n(j\tau)|}{|\mathbf{W}_n(0)|}\right\}\,. \tag{39}$$

Here, $\gamma_n(j\tau)$ is the nth order of the non-stationarity function.

The set of all memory functions $m_1(t), m_2(t), m_3(t), \ldots$ allows one to describe non-Markov processes and statistical memory effects in the considered non-stationary system. For the particular case we obtain a more simple form for the set of equations for the first three short memory functions, namely $(t=m\tau)$:

$$\begin{aligned}\frac{\Delta a(t)}{\Delta t} &= -\tau\Lambda_1 \sum_{j=0}^{m-1} m_1(j\tau)\left\{\frac{\gamma_1(j\tau)\gamma_1(t-j\tau)}{\gamma_1(t)}\right\}\\ &\quad\times a(t-j\tau) + \lambda_1 a(t)\,,\\ \frac{\Delta m_1(t)}{\Delta t} &= -\tau\Lambda_2 \sum_{j=0}^{m-1} m_2(j\tau)\left\{\frac{\gamma_2(j\tau)\gamma_2(t-j\tau)}{\gamma_2(t)}\right\}\\ &\quad\times m_1(t-j\tau) + \lambda_2 m_1(t)\,,\\ \frac{\Delta m_2(t)}{\Delta t} &= -\tau\Lambda_3 \sum_{j=0}^{m-1} m_3(j\tau)\left\{\frac{\gamma_3(j\tau)\gamma_3(t-j\tau)}{\gamma_3(t)}\right\}\\ &\quad\times m_2(t-j\tau) + \lambda_3 m_2(t)\,.\end{aligned} \tag{40}$$

Here the relaxation parameters Λ_1, Λ_2 and Λ_3 have already been determined and the non-stationarity functions $\gamma_n(t)$ have been introduced earlier. By analogy with Eq. (6) we can introduce a set of dynamic parameters of non-stationarity (PNS) for the arbitrary nth relaxation level

$$\Gamma_n(T,t) = 1-\gamma_n(t) = 1-\gamma_n(T,t)\,. \tag{41}$$

The whole set of values of dynamic PNS $\gamma_n(t)$ determines the broad spectrum of non-stationarity effects of the considered process.

The obtained equations are similar to the well known Zwanzig–Mori's kinetic equations [10,11,12,13,14,15,16,17,18] used in non-equilibrium statistical physics of condensed matters. Let us point out three essential distinctions of our Eqs. (40) from the results in [10,11,12]. In

Zwanzig–Mori's theory the key moment in the analysis of considered physical systems is the presence of a Hamiltonian and an operation of a statistical averaging carried out with the help of quantum density operator or classic Gibbs distribution function [33]. In our examined case, both the Hamiltonian and the distribution function are absent. There are exact classic or quantum equations of motion in physics; so Liouville's equation and Liouville's operator are useful in many applications. The motion of individual particles and whole statistic system is described by variables varying in continuous time. Therefore, for physical systems it is possible to use effectively the methods of integro-differential calculus, based on the mathematically accustomed (but from the physical point of view difficult for understanding) representation of infinitesimal variations of values of coordinates and time. By nature, the monitored time evolution of most complex systems is discrete. As well known, discretization is inherent in a wide variety both of classical and quantum complex systems. This forces us to abandon the concept of an infinite small values and continuity and instead turn to discrete-difference schemes. And, at last, the third feature is connected with incorporating the issue of non-stationary processes into our theory. The Zwanzig–Mori theory is typically applied only for stationary processes. Due to the introduction of normalized vectors of states and the use of the appropriate projection technique [13] our theory allows to take into account non-stationary processes as well. The latter ones can be described by the non-Markov kinetic equations together with the introduction of the set of non-stationarity functions.

The non-stationary theory [32] put forward here differs from the stationary case [31]. The external structure of the kinetic equations remains invariant; they represent the kinetic equations with memory. However, the functions and the parameters, which are included in these equations, appreciably differ from each other. As we already remarked above, non-stationarity effects enter both, in the functions $\gamma_n(t)$ and in spectral and kinetic parameters.

Correlation and Memory in Discrete Non-Markov Stochastic Processes Generated by Random Events

Here we shall find a chain of the kinetic interconnected finite-difference equations for a discrete correlation function $a(n)$ and memory functions $M_s(n)$ in the linear scale of events $E = \{\xi_1, \xi_2, \xi_3, \ldots, \xi_N\}$.

The Basic Assumptions and Concepts of the Theory of Discrete Non-Markov Stochastic Processes of the Events Correlations

As an example we shall consider the time variations of the total X-ray flux of an astrophysical object at a succession of events:

$$E = \{\xi_1, \xi_2, \xi_3, \ldots, \xi_k, \ldots, \xi_N\}\,, \qquad (42)$$

where ξ_i is an event, which occurs at time instant t_i, where $i = 1, \ldots, N$ counts the event number.

The average value $\langle E \rangle$, fluctuations $\delta\xi$ and dispersion σ^2 for the set of N events are obtained as:

$$\begin{aligned} \langle E \rangle &= \frac{1}{N}\sum_{i=1}^{N} \xi_i\,, \delta\xi_i = \xi_i - \langle E \rangle\,, \\ \sigma^2 &= \frac{1}{N}\sum_{i=1}^{N} \delta\xi_i^2 = \frac{1}{N}\sum_{i=1}^{N} \{\xi_i - \langle E \rangle\}^2\,. \end{aligned} \qquad (43)$$

According to [35,36,37,38], for the description of the dynamical properties of the studied system we introduce the correlation dependence of the discrete set of events (see Eq. (42)) using the CF:

$$a(n) = \frac{1}{(N-m)\,\sigma^2} \sum_{i=1}^{N-m} \delta\xi_i\, \delta\xi_{i+m}\,. \qquad (44)$$

Here $n = m\Delta n$, $\Delta n = 1$ is the discretization step. The function $a(n)$, which emerges in this way, is the "event" correlation function (ECF). The normalized ECF must obey the conditions of normalization and of the attenuation of correlation, i. e.: $\lim_{n\to 1} a(n) = 1$, $\lim_{n\to\infty} a(n) = 0$. We remark, however, that the second condition for the case the physical complex systems is typically not observed (at $N \gg 0$). It is necessary to note that in [18] the correlation function for the aftershock events has been introduced:

$$C(n+n_W, n_W) = \frac{[\langle t_{n+n_W}\, t_{n_W}\rangle - \langle t_{n+n_W}\rangle\langle t_{n_W}\rangle]}{\left(\sigma^2_{n+n_W}\sigma^2_{n_W}\right)^{1/2}}\,,$$

where the averages and the variance are given by

$$\begin{aligned} \langle t_m \rangle &= \frac{1}{N}\sum_{k=o}^{N-1} t_{m+k}\,, \\ \langle t_m\, t'_m \rangle &= \frac{1}{N}\sum_{k=o}^{N-1} t_{m+k}\, t'_{m+k}\,, \text{ and} \\ \sigma^2_m &= \langle t^2_m \rangle - \langle t_m \rangle^2\,, \end{aligned}$$

respectively.

By the direct analogy of [31,32,35] we use the finite-difference Liouville's equation of motion in the event scale for describing the evolution of discrete set of events Eq. (11), (13):

$$\frac{\Delta \xi_i(n)}{\Delta n} = i \widehat{L}(n,1)\, \xi_i(n)\,. \tag{45}$$

Here $\xi_i(n+1) = U(n+1,n)\,\xi_i(n)$, $U(n+1,n)$ is the "event" evolution operator. It determines the shift in linear event scale to one step Δn. The evolution operator $U(n+1,n)$ and Liouville's quasioperator $\widehat{L}(n,1)$ can be made explicit by writing: $\widehat{L}(n,1) = (i\Delta n)^{-1}\,(U(n+1,n) - 1)$.

Let's represent the set of values of the dynamical variable $\delta\xi_j = \delta\xi(j\Delta n)$, $j = 1,\ldots,N$ as the k-component vector of system state in linear Euclidean space:

a) the vector of initial state of studied complex system:

$$\mathbf{A}_k^1 = \{\delta\xi_1, \delta\xi_2, \delta\xi_3, \ldots, \delta\xi_k\}\,, \tag{46}$$

b) the vector of final system's state, which is shifted on the m events along the event scale:

$$\mathbf{A}_{m+k}^m = \{\delta\xi_{m+1}, \delta\xi_{m+2}, \delta\xi_{m+3}, \ldots, \delta\xi_{m+k}\}\,, \tag{47}$$

where $1 \le k \le N$. The vectors of initial and final states, which are submitted in a similar way, are very convenient for analyzing the dynamics of the observed discrete stochastic processes with the help of discrete non-Markov processes.

To represent the ECF in a more compact form, we use the expression for the scalar product of vectors $\langle \mathbf{A}_k^1 \cdot \mathbf{A}_{m+k}^m \rangle = \sum_{j=1}^{k} A_j^1 A_{m+j}^m$, and the Eqs. (64), (64) and (65) TS6:

$$a(n) = \frac{\langle \mathbf{A}_k^1(1)\, \mathbf{A}_{k+m}^m(n)\rangle}{\langle |\mathbf{A}_k^1(1)|^2\rangle}\,. \tag{48}$$

Construction of Chain of Finite-Difference Non-Markov Kinetic Equations for the Events Correlation

Let us consider the finite-difference Liouville's equation (Eq. (44)) for the vector of final system states:

$$\frac{\Delta \mathbf{A}_{m+k}^m(n)}{\Delta n} = i \widehat{L}(n,1)\, \mathbf{A}_{m+k}^m(n)\,. \tag{49}$$

We introduce the projection operator Π, which projects the final vector $\mathbf{A}_{m+k}^m(n)$ on the direction of initial vector, and also the orthogonal operator P. The operators Π and P possess the following properties: $\Pi = |\mathbf{A}_k^1(1)\rangle\langle \mathbf{A}_k^1(1)|/\langle |\mathbf{A}_k^1(1)|^2\rangle$, $\Pi^2 = \Pi$, $\mathrm{P} = 1 - \Pi$, $\mathrm{P}^2 = \mathrm{P}$, $\Pi \mathrm{P} = \mathrm{P}\Pi = 0$. They are idempotent and mutually complementary.

The initial ECF $a(n)$ (Eq. (48)) can be derived by means of projecting the vector of final states $\mathbf{A}_{m+k}^m(n)$ on the vector of initial state $\mathbf{A}_k^1(1)$:

$$\Pi \mathbf{A}_{m+k}^m(n) = \frac{\mathbf{A}_k^1(1)\langle \mathbf{A}_k^1(1)\, \mathbf{A}_{m+k}^m(n)\rangle}{\langle |\mathbf{A}_k^0|^2\rangle} = \mathbf{A}_k^1(1)\, a(n)\,. \tag{50}$$

The operators Π and P split Euclidean vector space $A(k)$ into two mutually orthogonal subspaces:

$$\begin{aligned} &A(k) = A'(k) + A''(k)\,, \quad A'(k) = \Pi A(k)\,,\\ &A''(k) = \mathrm{P}A(k)\,, \quad \mathbf{A}_{m+k}^m \in A(k)\,. \end{aligned} \tag{51}$$

As a result the finite-difference Liouville's Eq. (67) TS6 can be represented as a system of 2 equations into mutually orthogonal linear subspaces:

$$\frac{\Delta \mathbf{A}'(n)}{\Delta n} = i \widehat{L}_{11}\, \mathbf{A}'(n) + i \widehat{L}_{12}\, \mathbf{A}''(n)\,, \tag{52}$$

$$\frac{\Delta \mathbf{A}''(n)}{\Delta n} = i \widehat{L}_{21}\, \mathbf{A}'(n) + i \widehat{L}_{22}\, \mathbf{A}''(n)\,. \tag{53}$$

Here $\widehat{L}_{ij} = \Pi_i \widehat{L} \Pi_j$ are the matrix elements of Liouville's quasioperator:

$$\begin{aligned} &\widehat{L} = \widehat{L}_{11} + \widehat{L}_{12} + \widehat{L}_{21} + \widehat{L}_{22}\,,\\ &\widehat{L}_{11} = \Pi \widehat{L} \Pi\,, \quad \widehat{L}_{12} = \Pi \widehat{L} \mathrm{P}\,,\\ &\widehat{L}_{21} = \mathrm{P} \widehat{L} \Pi\,, \quad \widehat{L}_{22} = \mathrm{P} \widehat{L} \mathrm{P}\,. \end{aligned} \tag{54}$$

To solve the system of Eqs. (70), (71) TS6 we eliminate the non-reducible part, which contains $\mathbf{A}''(n)$ and derive the self-contained equation for the reducible part $\mathbf{A}'(n)$. In doing so we solve the Eq. (52) step-by-step and shall substitute the obtained solution into the Eq. (53). As a result we arrive at the closed kinetic equation:

$$\begin{aligned} \frac{\Delta \mathbf{A}'(n + m\Delta n)}{\Delta n} &= i \widehat{L}_{11}\, \mathbf{A}'(n + m\Delta n)\\ &\quad + i \widehat{L}_{12} \{1 + i\Delta n\, \widehat{L}_{22}\}^m\, \mathbf{A}''(n)\\ &\quad - \widehat{L}_{12} \sum_{j=1}^{m} \{1 + i\Delta n\, \widehat{L}_{22}\}^j\, \Delta n\\ &\quad \times \widehat{L}_{21}\, \mathbf{A}'(n + [m-j]\Delta n)\,. \end{aligned} \tag{55}$$

TS6 Please specify which equation(s) are meant here.

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By use of projection operators Π and P we found the closed finite-difference kinetic equation of non-Markov type for the initial ECF:

$$\frac{\Delta a(n)}{\Delta n} = i\lambda_1\, a(n) - \Delta n\, \Lambda_1 \sum_{j=1}^{m} M_1(j\Delta n)\, a(n - j\Delta n)\,. \tag{56}$$

As $\Delta n = 1$, solution of the last equation must be following:

$$a(n+1) = \{i\lambda_1 + 1\}\, a(n) - \Lambda_1 \sum_{j=1}^{m} M_1(j)\, a(n-j)\,. \tag{57}$$

Here λ_1 is the proper value of Liouville's quasioperator $\widehat{L}$, Λ_1 is the relaxation parameter, which dimension is square of frequency, $M_1(j\Delta n)$ is the normalized memory function of the first order:

$$\begin{aligned} \lambda_1 &= \frac{\langle A_k^1(1)\, \widehat{L}\, A_k^1(1)\rangle}{\langle |A_k^1(1)|^2\rangle}\,, \\ \Lambda_1 &= \frac{\langle A_k^1\, \widehat{L}_{12}\, \widehat{L}_{21}\, A_k^1(1)\rangle}{|A_k^1(1)|^2\rangle}\,, \\ M_1(j\Delta n) &= \frac{\langle A_k^1(1)\, \widehat{L}_{12}(1 + i\Delta n\, \widehat{L}_{22})^j\, \widehat{L}_{21}\, A_k^1(1)\rangle}{\langle A_k^1(1)\, \widehat{L}_{12}\, \widehat{L}_{21}\, A_k^1(1)\rangle}\,. \end{aligned}$$

To obtain the finite-difference kinetic equation for the normalized event memory function of first order and, further, for the higher $(s-1)$th orders as well, we have to repeat the foregoing procedure step-by-step. However, we shall make use of the Gram–Schmidt orthogonalization procedure [16]:

$$\langle \mathbf{W}_s \mathbf{W}_p\rangle = \delta_{sp}\, \langle |\mathbf{W}_s|^2\rangle\,. \tag{58}$$

Where δ_{sp} is a Kronecker's symbol. Now we shall derive the recurrence formula $\mathbf{W}_s = \mathbf{W}_s(n)$ for defining the set of the orthogonal dynamic variables:

$$\begin{aligned} \mathbf{W}_0 &= \mathbf{A}_k^1\,, \\ \mathbf{W}_1 &= \{i\widehat{L} - \lambda_1\}\mathbf{W}_0\,, \\ \mathbf{W}_2 &= \{i\widehat{L} - \lambda_2\}\mathbf{W}_1 - \Lambda_1 \mathbf{W}_0, \ldots \end{aligned} \tag{59}$$

According to the foregoing formulas we can introduce the succession of projection operators $\Pi_s = \Pi_1^{(s)}$ and the set of mutually complementary projectors $\mathrm{P}_s = 1 - \Pi_s$, which possess the following properties:

$$\begin{aligned} \Pi_s &= \frac{|\mathbf{W}_s\rangle\langle \mathbf{W}_s|}{\langle |\mathbf{W}_s|^2\rangle}\,, & \Pi_s^2 &= \Pi_s\,, \\ \mathrm{P}_s^2 &= \mathrm{P}_s\,, & \Pi_s\, \mathrm{P}_s &= \mathrm{P}_s\, \Pi_s = 0\,, \\ \Pi_s\, \Pi_p &= \delta_{sp}\, \Pi_s\,, & \mathrm{P}_s\, \mathrm{P}_p &= \delta_{sp}\, \mathrm{P}_s\,. \end{aligned}$$

Each of these operators pairs Π_s, P_s splits the corresponding Euclidean vector space $\mathbf{W}_s$ into the two mutual complementary subspaces: $W_s = W_s' + W_s'$, $W_s' = \Pi_s W_s$, $W_s'' = \mathrm{P}_s W_s$. Using the projection operator technique for the next orthogonal variables $\mathbf{W}_s$, we shall obtain the chain of interconnected kinetic finite-difference equations of the non-Markov type for the normalized correlation functions of the $(s-1)$th order:

$$\begin{aligned} \frac{\Delta M_1(n)}{\Delta n} &= i\,\lambda_2\, M_1(n) - \Lambda_2 \sum_{j=1}^{m} M_2(j)\, M_1(n-j)\,, \\ &\ldots\,, \\ \frac{\Delta M_{s-1}(n)}{\Delta n} &= i\,\lambda_s\, M_{s-1}(n) - \Lambda_s \sum_{j=1}^{m} M_{s-1}(j)\, M_s(n-j)\,. \end{aligned} \tag{60}$$

In these equations the normalized events memory function of the first order: $M_1(n) = \langle \mathbf{W}_1(1 + i\Delta n\widehat{L})^m \mathbf{W}_1\rangle / \langle |\mathbf{W}_1|^2\rangle$, memory function of the $(s-1)$th order: $M_{s-1}(n) = \langle \mathbf{W}_{s-1}(1 + i\Delta n\widehat{L})^m \mathbf{W}_{s-1}\rangle / \langle |\mathbf{W}_{s-1}|^2\rangle$, the proper value of the Liouville's quasioperator $\widehat{L}$: $\lambda_s = \langle \mathbf{W}_s \widehat{L} \mathbf{W}_s\rangle / \langle |\mathbf{W}_s|^2\rangle$ and the relaxation parameter $\Lambda_s = \langle |\mathbf{W}_s|^2\rangle / \langle |\mathbf{W}_{s-1}|^2\rangle$ are introduced.

The foregoing finite-difference kinetic Eqs. (60) present the generalization of the statistical theory [31,32,35] for the case of event correlations in discrete stochastic evolution of non-Hamilton complex systems.

Information Measures of Memory in Complex Systems

As an information measures of memory it is useful to apply different dimensionless quantifiers. As a first measure we use the frequency dependence of non-Markovity parameter. This measure was introduced in [31] and it is defined as:

$$\varepsilon_i(\nu) = \left\{\frac{\mu_{i-1}(\nu)}{\mu_i(\nu)}\right\}^{1/2}\,. \tag{61}$$

Here, $\mu_i(\nu)$ denotes the frequency power spectrum of memory function of the ist order $M_i(n)$: $\mu_i(\nu) = |\Delta n \sum_{n=1}^{N} M_i(n) \cos(2\pi n\nu)|^2$. The non-Markovity parameter $\varepsilon_i(\nu)$ along with the memory functions enables us to characterize quantitatively the statistical memory effects in discrete complex systems of various nature. Because the functions $\mu_i(\nu)$ exist for each of the ith levels of relaxation, we obtain the statistical spectrum of parameters: $\varepsilon_i(\nu)$, $i = 1, 2, 3, \ldots$.

Alternatively, a study of 'memory' in physiological time series for electroencephalographic (EEG) and magnetoencephalographic (MEG) signals, both of healthy subjects and patients (including epilepsy patients) has been based on the detrended-fluctuation analysis (DFA) [39,40].

The characterization of memory *per se* is based on a set of dimensionless statistical quantifiers which are capable for measuring the memory strength which is inherent to the complex dynamics.

According to [41] a second set an information memory measure can be constructed as follows:

$$\delta_i(\nu) = \left| \frac{\tilde{M}'_i(\nu)}{\tilde{M}'_{i+1}(\nu)} \right| .$$

Here, $\mu_i(\nu) = |\tilde{M}_i(\nu)|^2$ denotes the power spectrum of the corresponding memory function $M_i(t)$, $\tilde{M}'_i(\nu) = \mathrm{d}\tilde{M}_i(\nu)/\mathrm{d}\nu$ and $\tilde{M}_i(\nu)$ is the Fourier transform of the memory function $M_i(t)$. The measures $\varepsilon_i(\nu)$ are suitable for the quantification of the memory effects on a relative scale whereas the second set $\delta_i(\nu)$ proves to be useful for quantifying the amplification of relative memory effects occurring on different complexity levels. Both measures provide statistical criteria for comparison between the relaxation time scales and memory time scales of the process under consideration. For values obeying $\{\varepsilon, \delta\} \gg 1$ one can observe a complex dynamics characterized by the short-ranged temporal memory scales. In the memoryless limit these processes assume a δ-like memory with parameters $\varepsilon, \delta \to \infty$. When $\{\varepsilon, \delta\} > 1$ one deals with a situation with moderate memory strength, and the case where both $\varepsilon, \delta \sim 1$ typically constitutes a more regular and robust random process exhibiting strong memory features.

Manifestation of Strong Memory in Complex Systems

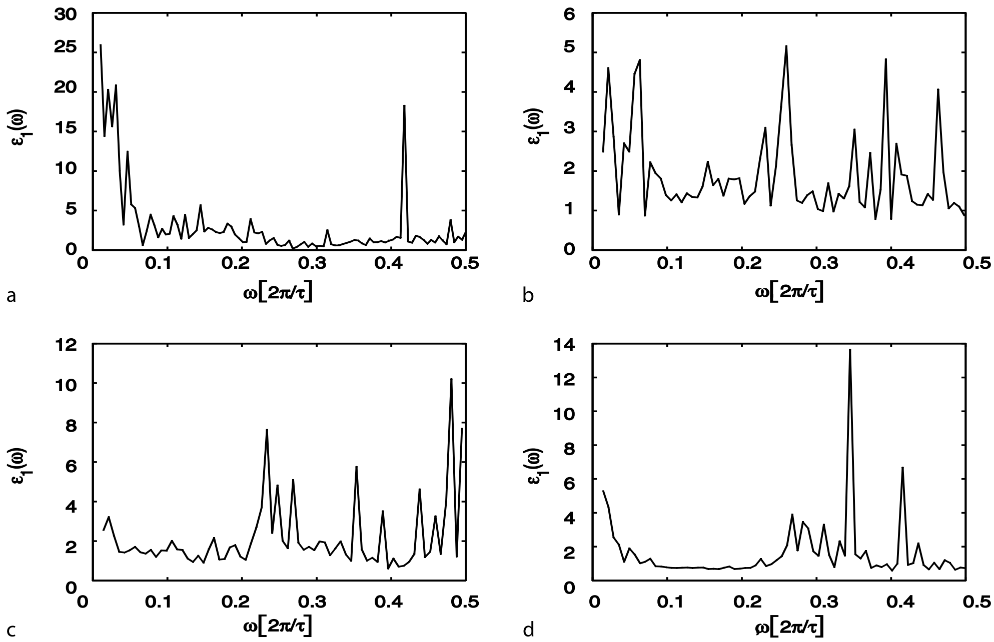
A fundamental role of the strong and weak memory in the functioning of the human organism and seismic phenomena can be illustrated by the example of some situations examined next. We will consider some examples of the time series for both living and for seismic systems. It is necessary to note that a comprehensive analysis of the experimental data includes the calculation and the presentation of corresponding phase portraits in some planes of the dynamic orthogonal variables, the autocorrelation time functions, the memory time functions and their frequency power spectra, etc. However, we start out by calculating two statistical quantifiers, characterizing two informational measures of memory: the parameters $\epsilon_1(\omega)$ and $\delta_1(\omega)$.

Figures 1 and 3 present the results of experimental data of pathological states of human cardiovascular systems (CVS). Figure 2 depicts the analysis for the seismic observation. Figures 4 and 5 indicate the memory effects for the patients with Parkinson disease (PD), and the last two Figs. 6, 7 demonstrate the key role of the strength of memory in the case of time series of patients suffering from photosensitive epilepsy which are contrasted with signals taken from healthy subjects. All these cases convincingly display the crucial role of the statistical memory in the functioning of complex (living and seismic) systems.

A characteristic role of the statistical memory can be detected from Fig. 1 for the typical representatives taken from patients from four different CVS-groups: (a) for healthy subject, (b) for a patient with rhythm driver migration, (c) for a patient after myocardial infarction (MI), (d) for a patient after MI with subsequent sudden cardiac death (SSCD). All these data were obtained from the short time series of the dynamics of RR-intervals from the electric signals of the human ECG's. It can be seen here that significant memory effects typically lead to the long-time correlations in the complex systems. For healthy we observe weak memory effects while and large values of the measure memory $\epsilon_1(\omega = 0) \approx 25$. The strong memory and the long memory time (approximately, 10 times more) are being observed with the help of 3 patient groups: with RDM (rhythm driver migration) (b), after MI (c) and after MI with SSCD (d).

Figure 2 depicts the strong memory effects presented in seismic phenomena. By a transition from the steady state of Earth ((a), (b) and (c)) to the state of strong earthquake (EQ) ((d), (e), and (f)) a remarkable amplification of memory effects is highly visible. The term amplification refers to the appearance of strong memory and the prolongation of the memory correlation time in the seismic system. Recent study show that discrete non-Markov stochastic processes and long-range memory effects play a crucial role in the behavior of seismic systems. An approach, permitting us to obtain an algorithm of strong EQ forecasting and to differentiate technogenic explosions from weak EQs, can be developed thereupon.

Figure 3 demonstrates an intensification of memory effects of one order at the transition from healthy people ((a), (b) and (c)) to patient suffering from myocardial infarction. The figures were calculated from the long time series of the RR-intervals dynamics from the human ECG's. The zero frequency values $\epsilon_1(\omega = 0)$ at $\omega = 0$ sharply reduced, approximately of the size of one order for patient as compared to healthy subjects.



Correlations in Complex Systems, Figure 1

Frequency spectrum of the first information measure of memory (first point in the statistical spectrum on non-Markovity parameter) $\varepsilon_1(\omega)$ for the fourth cardiac patient groups from the short time series of RR-intervals: healthy subject (a), patient with rhythm driver migration (RDM) (b), patient after myocardial infarction (MI) (c), and patient after MI with subsequent sudden cardiac death (SCD) (d). The frequency is marked in terms of units of τ^{-1}. All spectra reveal the miscellaneous faces of statistical memory's strength. For the healthy subject one can see Markov effects and weak memory. For other three cases of cardiac diseases we note the diverse displays of strong memory. The strong memory has been accompanied by the spikes of the weak memory: for RDM on the all frequency regions, for patient with MI for the middle and high frequencies and for patient after MI with SSCD only for high frequencies. From Fig. 7 in [104]

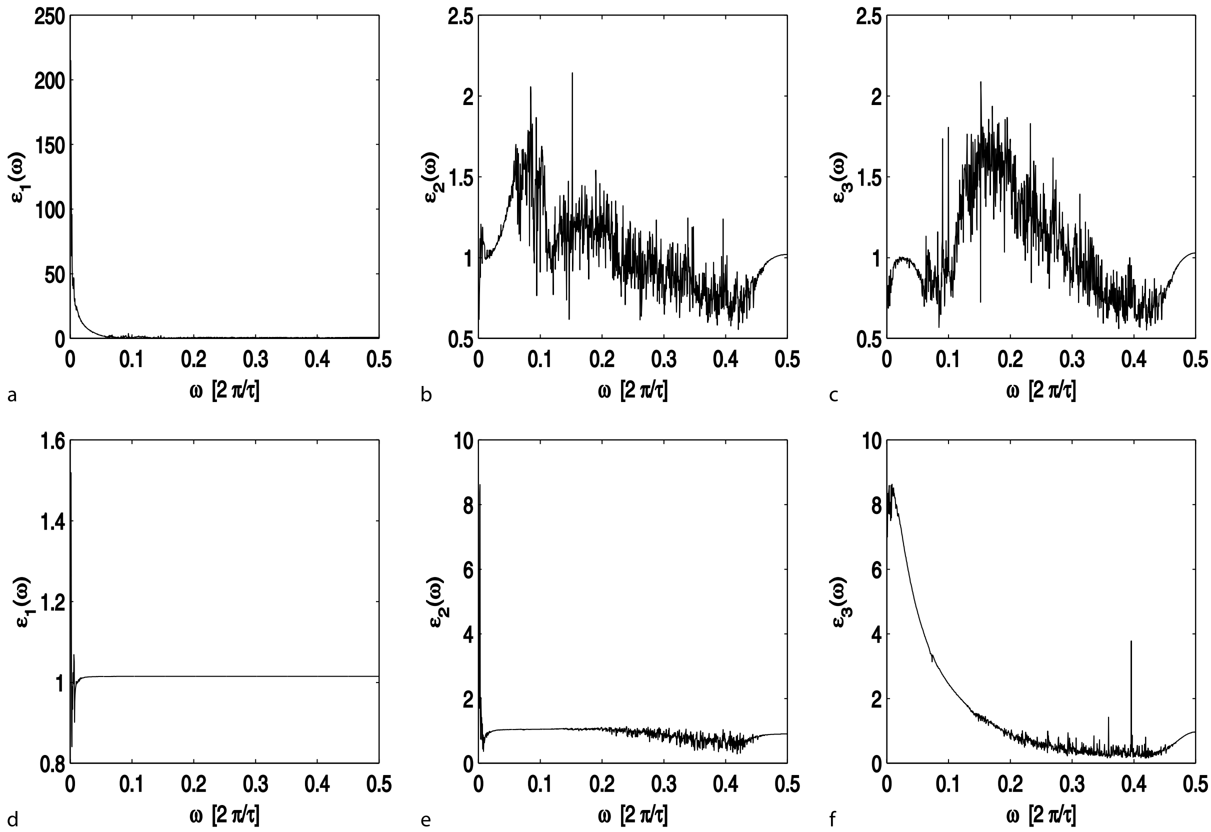
Figures 4 and 5 illustrate the behavior for patients with Parkinson's disease. Figure 4 shows time recording of the pathological tremor velocity in the left index finger of a patient with Parkinson's disease (PD) for eight diverse pathological cases (with or without medication, with or without deep brain stimulation (DBS), for various DBS, medication and time conditions). Figure 5, arranged in accordance with these conditions, displays a wide variety of the memory effects in the treatment of PD's patients. Due to the large impact of memory effects this observation permits us to develop an algorithm of exact diagnosis of Parkinson's disease and a calculation of the quantitative parameter of the quality of treatment. A physical role of the strong and long memory correlation time enables us to extract a vital information about the states of various patient on basis of notions of correlation and memory times.

According to Figs. 6 and 7 specific information about the physiological mechanism of photosensitive epilepsy (PSE) was obtained from the analysis of the strong memory effects via the registration the neuromagnetic responses in recording of magnetoencephalogram (MEG) of the human brain core. Figure 6 presents the topographic dependence of the first level of the second memory measure $\delta_1(\omega = 0; n)$ for the healthy subjects in the whole group (upper line) vs. patients (lower line) for red/blue combination of the light stimulus. This topographic dependence of $\varepsilon_1(\omega = 0; n)$ depicted in Fig. 6 clearly demonstrates the existence of long-range time correlation. It is accompanied by a sharp increase of the role of the statistical memory effects in the all MEG's sensors with sensor numbers $n = 1, 2, \ldots, 61$ of the patient with PSE in comparison with healthy peoples. A sizable difference between the healthy subject and a subject with PSE occurs.

To emphasize the role of strong memory one can continue studying the topographic dependence in terms of the novel informational measure, the index of memory, defined as:

$$\nu(n) = \frac{\delta_1^{\text{healthy}}(0; n)}{\delta_1^{\text{patient}}(0; n)}\,, \qquad (62)$$

see in Fig. 7.

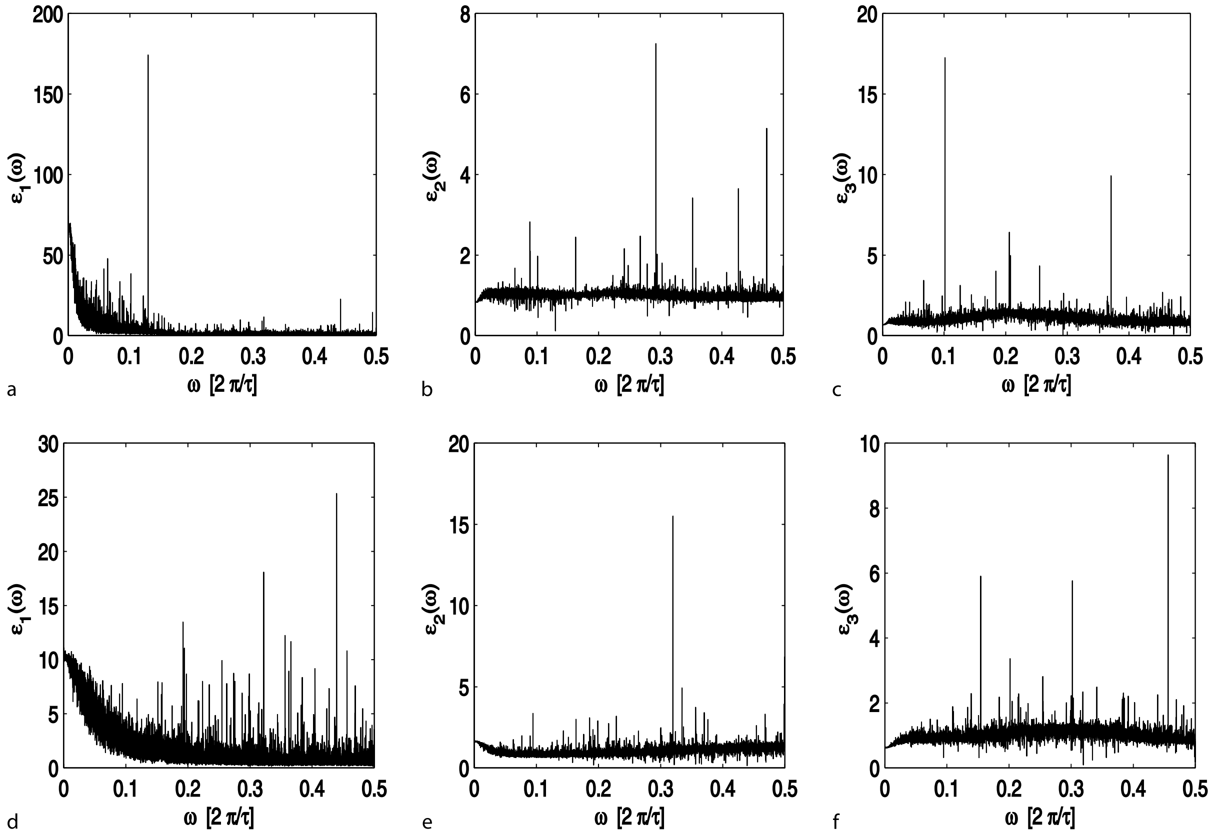


Correlations in Complex Systems, Figure 2

Frequency spectra of the first three points of the first measure of memory (non-Markovity parameters) $\varepsilon_1(\omega)$, $\varepsilon_2(\omega)$, and $\varepsilon_3(\omega)$ for the seismic phenomena: a, b, c long before the strong Earthquake (EQ) for the steady state of Earth and d, e, f during the strong EQ. Markov and quasi-Markov behavior of seismic sign[illegible]ith manifestation of the weak memory is observed only for ε_1 in state before the strong EQ. All remaining cases b, c, d and d T[illegible]ate to non-Markov processes. Strong non-Markovity and strong memory is typical for case d (state during the strong EQ). In behavior of $\varepsilon_2(\omega)$ and $\varepsilon_3(\omega)$ one can see a transition from quasi-Markovity (at low frequencies) to strong non-Markovity (at high frequencies). From Fig. 6 in [105]

This measure quantifies the detailed memory effects in the individual MEG sensors of the patient with PSE versus the healthy group. A sharp increase of the role of the memory effects in the stochastic behavior of the magnetic signals is clearly detected in sensor numbers $n = 10, 46, 51, 53$ and 59. The observed points of MEG sensors locate the regions of a protective mechanism against PSE in a human organism: frontal (sensor 10), occipital (sensors 46, 51 and 53) and right parietal (sensor 59) regions. The early activity in these sensors may reflect a protective mechanism suppressing the cortical hyperactivity due to the chromatic flickering.

We remark that some early steps towards understanding the normal and various catastrophical states of complex systems have already been taken in many fields of science such as cardiology, physiology, medicine, neurology, clinical neurophysiology, neuroscience, seismology and so forth. With the underlying systems showing fractal and complicated spatial structures numerous studies applying the linear and nonlinear time series analysis to various complex systems have been discussed by many authors. Specifically the results obtained shows evidence of the significant nonlinear structure evident in the registered signals in the control subjects, whereas nonlinearity for the patients and catastrophical states were not detected. Moreover the couplings between distant parts and regions were found to be stronger for the control subjects. These prior findings are leading to the hypothesis that the real normal complex systems are mostly equipped with significantly nonlinear subsystems reflecting an inherent mechanism which stems against a synchronous excitation vs. outside impact or inside disturbances. Such nonlinear mechanisms are likely absent in the occurrence of catastrophical or pathological states of the complex systems.



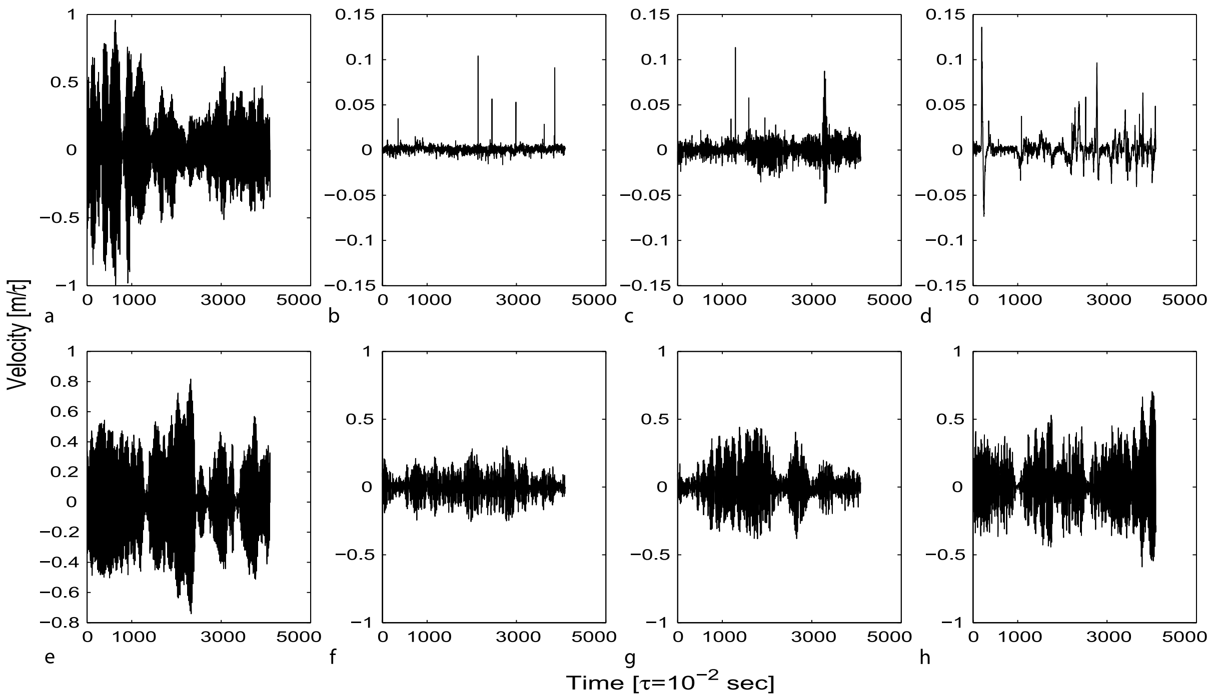
Correlations in Complex Systems, Figure 3

The frequency dependence of the first three points of non-Markovity parameter (NMP) for the healthy person (a), (b), (c) and patient after myocardial infarction (MI) (d), (e), (f) from the time dynamics of RR-intervals of human ECG's for the case of the long time series. In the spectrum of the first point of NMP $\varepsilon_1(\omega)$ there is an appreciable low-frequency (long time) component, which concerns the quasi-Markov processes. Spectra NMP $\varepsilon_2(\omega)$ and NMP $\varepsilon_3(\omega)$ fully comply with non-Markov processes within the whole range of frequencies. From Fig. 6 in [106]

From the physical point of view our results can be used as a toolbox for testing and identifying the presence or absence of various memory effects as they occur in complex systems. The set of our memory quantifiers is uniquely associated with the appearance of memory features in the chaotic behavior of the observed signals. The registration of the behavior belonging to these indicators, as elucidated here, is of beneficial use for detecting the catastrophical or pathological states in the complex systems. There exist alternative quantifiers of different nature as well, such as the Lyapunov's exponent, Kolmogorov–Sinai entropy, correlation dimension, etc., which are widely used in nonlinear dynamics and relevant applications. In the present context, we have found out that the employed memory measures are not only convenient for the analysis but are also ideally suitable for the identification of anomalous behavior occurring in complex systems. The search for other quantifiers, and foremost, the ways of optimization of such measures when applied to the complex discrete time dynamics presents a real challenge. Especially this objective is met when attempts are made towards the identification and quantification of functioning in complex systems. This work presents initial steps towards the understanding of basic foundation of anomalous processes in complex systems on the basis of a study of the underlying memory effects and connected with this, the occurrence of long lasting correlations.

Some Perspectives on the Studies of Memory in Complex Systems

Here we present a few outlooks on the fundamental role of statistical memory in complex systems. This involves the issue of studying cross-correlations. The statistical theory of stochastic dynamics of cross-correlation can be created on the basis of the mentioned formalism of projection



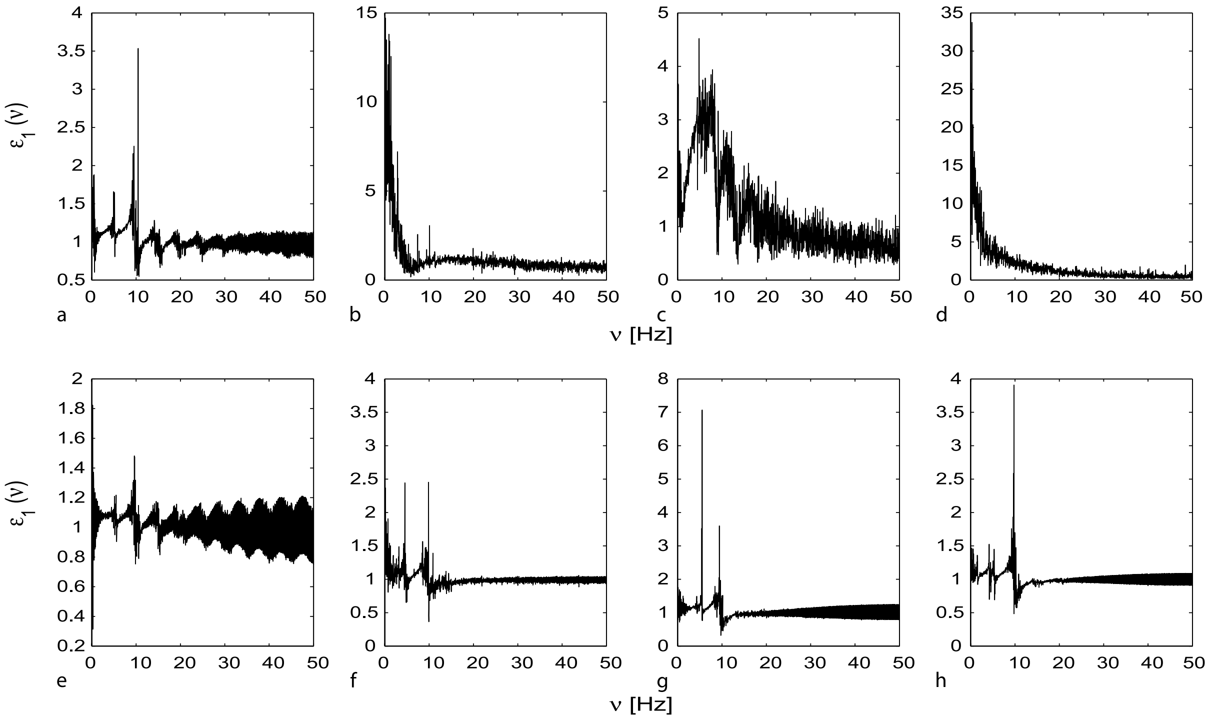
Correlations in Complex Systems, Figure 4
Pathological tremor velocity in the left index finger of the sixth patient with Parkinson's disease (PD). The registration of Parkinsonian tremor velocity is carried out for the following conditions: a "OFF-OFF" condition (no any treatment), b "ON-ON" condition (using deep brain stimulation (DBS) by electromagnetic stimulator and medicaments), c "ON-OFF" condition (DBS only), d "OFF-ON" condition (medicaments (L-Dopa) only), e–h the "15 OFF", "30 OFF", "45 OFF", "60 OFF" conditions – the patient's states 15 (30, 45, 60) minutes after the DBS is switched off, no treatment. Let's note the scale of the pathological tremor amplitude (see the vertical scale). Such representation of the time series allows us to note the increase or the decrease of pathological tremor. From Fig. 1 in [107]

operators technique in the linear space of random variables. As a result we obtain the cross-correlation memory functions (MF's) revealing the statistical memory effects in complex systems. Some memory quantifiers will appear simultaneously which will reflect cross-correlation between different parts of CS. Cross-correlation MF's can be very useful for the analysis of the weak and strong interactions, signifying interrelations between the different groups of random variables in CS. Besides that the cross-correlation can be important for the problem of phase synchronization, which can find a unique way of studying of synchronization phenomena in CS that has a special importance when studying aspects of brain and living systems dynamics.

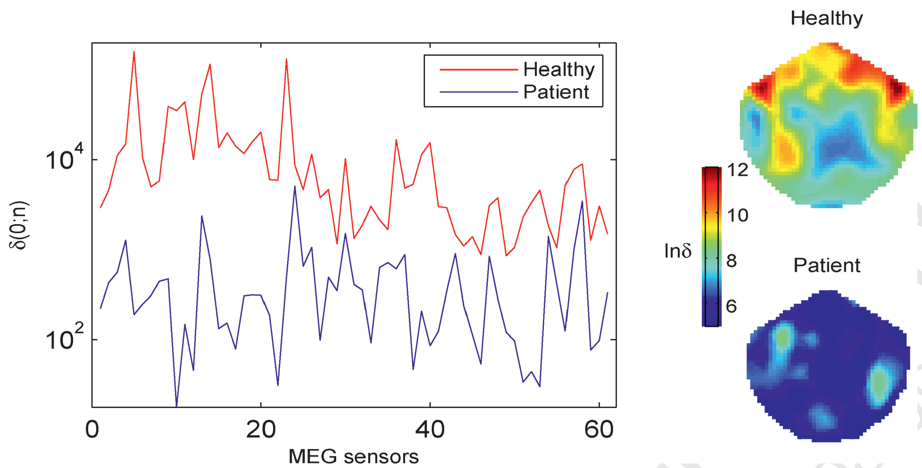
Some additional information about the strong and weak memory effects can be extracted from the observation of correlation in CS in the random event's scales. Similar effects are playing a crucial role in the differentiation between stochastic phenomena within astrophysical systems, for example, in galaxies, pulsars, quasars, microquasars, lacertides, black holes, etc. One of the most important area of application of developed approach is a bispectral and polyspectral analysis for the diverse CS. From the mathematical point of view a correct definition of the spectral properties in the functional space of random functions is quite important. A variety of MF's arises in the quantitative analysis of the fine details of memory effects in a nonlinear manner. The quantitative control of the treatment quality in the diverse areas of medicine and physiology may be one of the important biomedical application of the manifestation of the strong memory effects.

These and other features of memory effects in CS call for an advanced development of brain studies on the basis of EEG's and MEG's data, cardiovascular, locomotor and respiratory human systems, in the development of the control system of information flows in living systems. An example is the prediction of strong EQ's and the clear differentiation between the occurrence of weak EQ's and the technogenic explosions, etc.

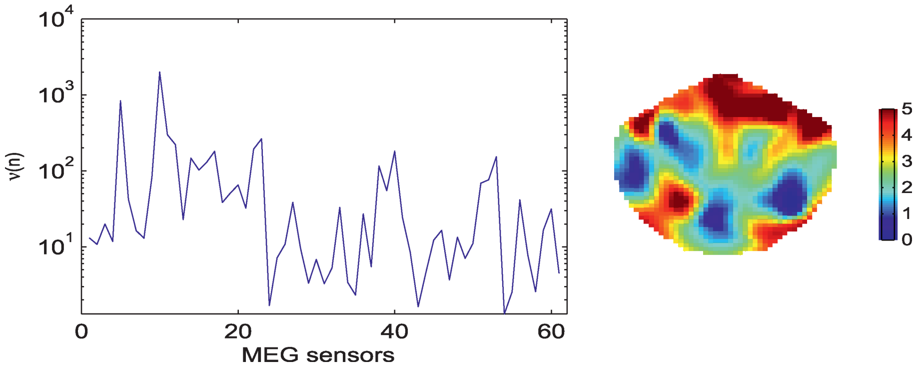
In conclusion, we hope that the interested reader becomes invigorated by this presentation of correlation and memory analysis of the inherent nonlinear system



Correlations in Complex Systems, Figure 5
The frequency dependence of the first point of the non-Markovity parameter $\varepsilon_1(\nu)$ for pathological tremor velocity in the patient. As an example, the sixth patient with Parkinson's disease is chosen. The figures are submitted according to the arrangement of the initial time series. The characteristic low-frequency oscillations are observed in frequency dependence (a, e–h), which get suppressed under medical influence (b–d). The non-Markovity parameter reflects the Markov and non-Markov components of the initial time signal. The value of the parameter on zero frequency $\varepsilon_1(0)$ reflects the total dynamics of the initial time signal. The maximal values of parameter $\varepsilon_1(0)$ correspond to small amplitudes of pathological tremor velocity. The minimal values of this parameter are characteristic of significant pathological tremor velocities. The comparative analysis of frequency dependence $\varepsilon_1(\nu)$ allows us to estimate the efficiency of each method of treatment. From Fig. 5 in [107]



Correlations in Complex Systems, Figure 6
The topographic dependence of the first point of the second measure of memory $\delta_1(\omega = 0; n)$ for the healthy on average in the whole group (*upper line*) vs. patient (*lower line*) for R/B combination of the light stimulus. One can note the singular weak memory effects for the healthy on average in sensors with No. 5, 23, 14, 11 and 9



Correlations in Complex Systems, Figure 7
The topographic dependence of the memory index $\nu(n) = \nu_1(n; 0)$ for the the whole group of healthy on average vs. patient for an R/B combination of the light stimulus. Strong memory in patient vs. healthy appears clearly in sensors with No. 10, 5, 23, 40 and 53

dynamics of varying complexity. He can find further details how significant memory effects typically cause long time correlations in complex systems by inspecting more closely some of the published items in [42–103].

There are the relationships between standard fractional and polyfractal processes and long-time correlation in complex systems, which were explained in [39,40,44,45, 46,49,53,54,60,62,64,76,79,83,84,94] in detail.

Example of using the Hurst exponent over time for testing the assertion that emerging markets are becoming more efficient can be found in [51].

While over 30 measures of complexity have been proposed in the research literature one can distinguish [42,55, 66,81,89,99] with the specific designation of long-time correlation and memory effects.

Papers [48,57] are focused on long range correlation processes that are nonlocal in time and whence show memory effects.

The statistical characterization of the nonstationarities in real-world time series is an important topic in many fields of research and some numerous methods of characterizing nonstationary time series were offered in [59, 65,84].

Long-range correlated time series have been widely used in [52,61,63,68,74] for the theoretical description of diverse phenomena.

Example of the study an anatomy of extreme events in a complex adaptive system can be found in [67].

Approaches for modeling long-time and long-range correlation in complex systems from time series are investigated and applied to different examples in [50,56,69,70, 73,75,80,82,86,100,101,102].

Detecting scale invariance and its fundamental relationships with statistical structures is one of the most relevant problems among those addressed correlation analysis [47,71,72,91].

Specific long-range correlation in complex systems are the object of active research due to its implications in the technology of materials and in several fields of scientific knowledge with the use of quantified histograms [78], decrease of chaos in heart failure [85], scaling properties of ECG's signals fluctuations [87] , transport properties in correlated systems [88] etc.

It is demonstrated in [43,92,93] how ubiquity of the long-range correlations is apparent in typical and exotic complex statistical systems with application to biology, medicine, economics and to time clustering properties [95,98].

The scale-dependent wavelet and spectral measures for assessing cardiac dysfunction have been used in [97].

In recent years the study of an increasing number of natural phenomena that appear to deviate from standard statistical distributions has kindled interest in alternative formulations of statistical mechanics [58,101].

At last, papers [77,90] present the samples of the deep and multiple interplay between discrete and continuous long-time correlation and memory in complex systems and the corresponding modeling the discrete time series on the basis of physical Zwanzig–Mori's kinetic equation for the Hamilton statistical systems.

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